Stokes's Theorem



Idea: Green's Theorem for surfaces in 3-space.

Recall (Green's Theorem): Let $D \subseteq \mathbb{R}^2$ be compact and suppose ∂D is a simple, closed, piecewise-smooth curve. If P and Q have continuous partial derivatives on an open region containing D, then

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Observe that $\operatorname{curl}(P\vec{i} + Q\vec{j}) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$.

Moreover if ∂D is parameterized by $\vec{r}(t) = \langle x(t), y(t), 0 \rangle$ on $I \subseteq \mathbb{R}$,

$$\begin{aligned}
\iint_D \operatorname{curl}(P\vec{i} + Q\vec{j}) \cdot d\vec{S}
&= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P\,dx + Q\,dy \\
&= \int_I \Big(P(\vec{r}(t))\, x'(t) + Q(r(t))\, y'(t) \Big)\, dt \\
&= \int_I \Big(P(\vec{r}(t))\,\vec{i} + Q(\vec{r}(t))\,\vec{j} \Big) \cdot \vec{r}\,'(t)\, dt \\
&= \int_{\partial D} (P\vec{i} + Q\vec{j}) \cdot d\vec{r}
\end{aligned}$$

Point: Green's Theorem can be expressed with surface integrals of (boring) vector fields over (boring) surfaces...

Q: Can we make this less boring? **A:** Yes!

Prop (Stokes' Theorem): Let S be an orientable smooth surface in $\mathbb{R}^3$ with ∂S a closed, piecewise-smooth single-component curve. If $\vec{F}$ is a vector field on $\mathbb{R}^3$ with continuous partial derivatives on an open region containing S, then $\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$.

pf (Via Green's Theorem): Smoothly parameterize S by $\vec{S}: D \to \mathbb{R}^3$ s.t.

1. ∂D is a piecewise-smooth simple closed curve.
2. $\vec{r}: \partial D \to \partial S$ parameterizes ∂S. ← on ∂D, "$\vec{r} = \vec{S}$"

By the multivariate chain rule: $\frac{d}{dt}\left[\vec{r}(u(t), v(t))\right] = \frac{\partial \vec{r}}{\partial u} u'(t) + \frac{\partial \vec{r}}{\partial v} v'(t)$.

Now we can express our line integral as follows:

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_{\partial D} \vec{F} \cdot \frac{\partial \vec{r}}{\partial u} du + \vec{F} \cdot \frac{\partial \vec{r}}{\partial v} dv$$

Now define functions $P(u,v) = \vec{F} \cdot \frac{\partial \vec{r}}{\partial u}$, $Q(u,v) = \vec{F} \cdot \frac{\partial \vec{r}}{\partial v}$.

By our assumptions, Green's Theorem applies! We obtain

$$\int_{\partial D} \vec{F} \cdot \frac{\partial \vec{r}}{\partial u} du + \vec{F} \cdot \frac{\partial \vec{r}}{\partial v} dv = \int_{\partial D} P\, du + Q\, dv = \iint_D \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) dA$$

?//

Now, $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \iint_D \operatorname{curl}(\vec{F}) \cdot (\vec{S}_u \times \vec{S}_v)\, dA$, so we will finish the proof if we can show the following equality:

$$\operatorname{curl}(\vec{F}) \cdot (\vec{S}_u \times \vec{S}_v) = \frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}$$

We calculate as follows:

Firstly, $\frac{\partial Q}{\partial u} = \frac{\partial}{\partial u}\left[\vec{F}\cdot\frac{\partial \vec{r}}{\partial v}\right] = \frac{\partial \vec{F}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v} + \vec{F}\cdot\frac{\partial^2 \vec{r}}{\partial u \partial v}$

and $\frac{\partial P}{\partial v} = \frac{\partial}{\partial v}\left[\vec{F}\cdot\frac{\partial \vec{r}}{\partial u}\right] = \frac{\partial \vec{F}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial u} + \vec{F}\cdot\frac{\partial^2 \vec{r}}{\partial v \partial u}$.

Note $\frac{\partial^2 \vec{r}}{\partial u \partial v} = \frac{\partial^2 \vec{r}}{\partial v \partial u}$ by Clairaut's Theorem (guaranteed by our assumptions).

Next we apply the chain rule to compute both

$$\frac{\partial \vec{F}}{\partial u} = \frac{\partial \vec{F}}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \vec{F}}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \vec{F}}{\partial z}\frac{\partial z}{\partial u}$$

and $$\frac{\partial \vec{F}}{\partial v} = \frac{\partial \vec{F}}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \vec{F}}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial \vec{F}}{\partial z}\frac{\partial z}{\partial v}.$$

Thus we obtain the following formula:

$$\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} = \left(\frac{\partial \vec{F}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v} + \vec{F}\cdot\frac{\partial^2 \vec{r}}{\partial u \partial v}\right) - \left(\frac{\partial \vec{F}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial u} + \vec{F}\cdot\frac{\partial^2 \vec{r}}{\partial v \partial u}\right)$$

$$= \frac{\partial \vec{F}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v} - \frac{\partial \vec{F}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial u}$$

$$= \left(\frac{\partial \vec{F}}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \vec{F}}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \vec{F}}{\partial z}\frac{\partial z}{\partial u}\right)\cdot\frac{\partial \vec{r}}{\partial v} - \left(\frac{\partial \vec{F}}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \vec{F}}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial \vec{F}}{\partial z}\frac{\partial z}{\partial v}\right)\cdot\frac{\partial \vec{r}}{\partial u}$$

$$= \frac{\partial \vec{F}}{\partial x}\cdot\left(\frac{\partial x}{\partial u}\frac{\partial \vec{r}}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial \vec{r}}{\partial u}\right) + \frac{\partial \vec{F}}{\partial y}\cdot\left(\frac{\partial y}{\partial u}\frac{\partial \vec{r}}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial \vec{r}}{\partial u}\right) + \frac{\partial \vec{F}}{\partial z}\cdot\left(\frac{\partial z}{\partial u}\frac{\partial \vec{r}}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial \vec{r}}{\partial u}\right)$$

On the other hand, we compute (noting $\vec{r}$ is $\vec{S}$ restricted to ∂S):

$$\operatorname{curl}(\vec{F})\cdot(\vec{S}_u\times\vec{S}_v) = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, -\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right), \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right\rangle\cdot(\vec{S}_u\times\vec{S}_v)$$

$\vec{F} = \langle F_x, F_y, F_z\rangle$

$$= \begin{vmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} & \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} & \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}$$

$$- \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\frac{\partial z}{\partial u}\frac{\partial y}{\partial v} - \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\frac{\partial x}{\partial u}\frac{\partial z}{\partial v}$$

$$= \frac{\partial F_x}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial x}{\partial u}\right) + \frac{\partial F_y}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) + \frac{\partial F_z}{\partial x}\left(\frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\right)$$

$$+ \frac{\partial F_x}{\partial y}\left(\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial u}\right) + \frac{\partial F_y}{\partial y}\left(\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial y}{\partial u}\right) + \frac{\partial F_z}{\partial y}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)$$

$$+ \frac{\partial F_x}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) + \frac{\partial F_y}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial y}{\partial u}\right) + \frac{\partial F_z}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial \vec{F}}{\partial x} \cdot \left\langle \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial x}{\partial u}, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}, \frac{\partial x}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial z}{\partial u}\right\rangle$$

$$+ \frac{\partial \vec{F}}{\partial y} \cdot \left\langle \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial y}{\partial u}, \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right\rangle$$

$$+ \frac{\partial \vec{F}}{\partial z} \cdot \left\langle \frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}, \frac{\partial z}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial z}{\partial u}\right\rangle$$

$$= \frac{\partial \vec{F}}{\partial x} \cdot \left(\frac{\partial x}{\partial u}\frac{\partial \vec{r}}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial \vec{r}}{\partial u}\right) + \frac{\partial \vec{F}}{\partial y} \cdot \left(\frac{\partial y}{\partial u}\frac{\partial \vec{r}}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial \vec{r}}{\partial u}\right) + \frac{\partial \vec{F}}{\partial z} \cdot \left(\frac{\partial z}{\partial u}\frac{\partial \vec{r}}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial \vec{r}}{\partial u}\right)$$

Hence we put this all together to obtain the result:

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_{\partial D} \vec{F} \cdot \frac{\partial \vec{r}}{\partial u}\, du + \vec{F} \cdot \frac{\partial \vec{r}}{\partial v}\, dv$$

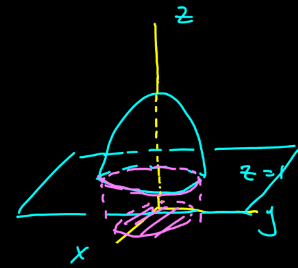
$$= \iint_D \left(\frac{\partial}{\partial u}\left[\vec{F} \cdot \frac{\partial \vec{r}}{\partial v}\right] - \frac{\partial}{\partial v}\left[\vec{F} \cdot \frac{\partial \vec{r}}{\partial u}\right]\right) dA$$

$$= \iint_D \left(\frac{\partial \vec{F}}{\partial x} \cdot \left(\frac{\partial x}{\partial u}\frac{\partial \vec{r}}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial \vec{r}}{\partial u}\right) + \frac{\partial \vec{F}}{\partial y} \cdot \left(\frac{\partial x}{\partial u}\frac{\partial \vec{r}}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial \vec{r}}{\partial u}\right) + \frac{\partial \vec{F}}{\partial z} \cdot \left(\frac{\partial x}{\partial u}\frac{\partial \vec{r}}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial \vec{r}}{\partial u}\right)\right) dA$$

$$= \iint_D \mathrm{Curl}(\vec{F}) \cdot (\vec{S}_u \times \vec{S}_v)\, dA = \iint_S \mathrm{Curl}(\vec{F}) \cdot d\vec{S} \qquad \square$$

Ex: Compute $\int_{\partial S} \vec{F}\cdot d\vec{r}$ and $\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}$ separately for $\vec{F} = \langle -2yz, y, 3x\rangle$ and $S = \{(x,y,z) : \underline{z = 5-x^2-y^2} \text{ and } \underline{z \geq 1}\}$.

Sol: First compute $\int_{\partial S}\vec{F}\cdot d\vec{r}$. Parametrize ∂S by



$$1 = z = 5 - x^2 - y^2 \iff x^2 + y^2 = 4 \quad \text{and} \quad z = 1$$

So ∂S is parameterized by

$$\vec{r}(\theta) = \langle 2\cos(\theta), 2\sin(\theta), 1\rangle \quad \text{on} \quad 0 \leq \theta \leq 2\pi$$

$$\vec{r}\,'(\theta) = \langle -2\sin(\theta), 2\cos(\theta), 0\rangle$$

$$\vec{F}(\vec{r}(\theta)) = \langle -2\cdot 2\sin(\theta)\cdot 1, \; 2\sin(\theta), \; 3\cdot 2\cos(\theta)\rangle$$
$$= \langle -4\sin(\theta), 2\sin(\theta), 6\cos(\theta)\rangle$$

$$\therefore \int_{\partial S} \vec{F}\cdot d\vec{r} = \int_{\theta=0}^{2\pi} \left(8\sin^2(\theta) + 4\sin(\theta)\cos(\theta)\right) d\theta$$

$$= 4\int_{\theta=0}^{2\pi} \left(1 - \cos(2\theta) + \sin(\theta)\cos(\theta)\right) d\theta$$

$$= 4\left[\theta - \tfrac{1}{2}\sin(2\theta) + \tfrac{1}{2}\sin^2(\theta)\right]_{\theta=0}^{2\pi} = 4\left((2\pi - 0) - 0\right) = 8\pi$$

Next compute $\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S}$. $S = \{(x,y,z) : z = 5 - x^2 - y^2, \; z \geq 1\}$

$$\vec{S}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 5 - r^2\rangle \quad \text{on} \quad D = \{(r,\theta) : 0 \leq r \leq 2, \; 0 \leq \theta \leq 2\pi\}$$

$$\left(\text{Recall: } \iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S} = \iint_D \operatorname{curl}(\vec{F})\cdot(\vec{S}_r \times \vec{S}_\theta)\, dA\right)$$

$$\operatorname{Curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2yz & y & 3x \end{vmatrix} = \left\langle \frac{\partial}{\partial y}[3x] - \frac{\partial}{\partial z}[y], -\left(\frac{\partial}{\partial x}[3x] - \frac{\partial}{\partial z}[-2yz]\right), \frac{\partial}{\partial x}[y] - \frac{\partial}{\partial y}[-2yz]\right\rangle$$

$$= \langle 0, -2y - 3, 2z\rangle$$

$$= \langle 0, -2r\sin(\theta) - 3, 10 - 2r^2\rangle$$

$$\therefore \operatorname{Curl}(\vec{F}) \cdot (\vec{S}_r \times S_\theta) = \begin{vmatrix} 0 & -2r\sin(\theta) - 3 & 10 - 2r^2 \\ \cos(\theta) & \sin(\theta) & -2r \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix}$$

$$= 0 \begin{vmatrix} \sin(\theta) & -2r \\ r\cos(\theta) & 0 \end{vmatrix} - \left(-2r\sin(\theta) - 3\right) \begin{vmatrix} \cos(\theta) & -2r \\ -r\sin(\theta) & 0 \end{vmatrix} + \left(10 - 2r^2\right) \begin{vmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{vmatrix}$$

$$= 0 + \left(2r\sin(\theta) + 3\right)\left(-2r^2\sin(\theta)\right) + \left(10 - 2r^2\right)\left(r\cos^2(\theta) + r\sin^2(\theta)\right)$$

$$= -4r^3\sin^2(\theta) - 6r^2\sin(\theta) + 10r - 2r^3$$

$$\therefore \iint_S \operatorname{Curl}(\vec{F}) \cdot d\vec{S} = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left(-4r^3\sin^2(\theta) - 6r^2\sin(\theta) + 10r - 2r^3\right) dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[-r^4\sin^2(\theta) - 2r^3\sin(\theta) + 5r^2 - \tfrac{1}{2}r^4\right]_{r=0}^{2} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(-16\sin^2(\theta) - 16\sin(\theta) + 20 - \tfrac{1}{2}\cdot 16 - 0\right) d\theta$$

$$= \int_{\theta=0}^{2\pi} -8\left(1 - \cos(2\theta)\right) - 16\sin(\theta) + 12\, d\theta$$

$$= \left[-8\left(\theta - \tfrac{1}{2}\sin(2\theta)\right) + 16\cos(\theta) + 12\theta\right]_{\theta=0}^{2\pi}$$

$$= \left[4\theta + 4\sin(2\theta) + 16\cos(\theta)\right]_{\theta=0}^{2\pi}$$

$$= 4(2\pi - 0) + 4(0 - 0) + 16(1 - 1) = 8\pi$$

Thus we observe $\iint_S \operatorname{curl}(\vec{F}) \cdot d\vec{S} = 8\pi = \int_{\partial S} \vec{F} \cdot d\vec{r}$ //

Ex: Compute $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F} = \langle -y^2, x, z^2 \rangle$ and C the curve of intersection of $y+z=2$ and $x^2+y^2=1$ in $\mathbb{R}^3$.

Sol: Note that C is the boundary of the surface

$$S = \{(x,y,z) : y+z=2 \text{ and } x^2+y^2 \le 1\}$$

NB: $C = \partial S$



$$\vec{S}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 2 - r\sin(\theta) \rangle \quad \text{for } (r,\theta) \in [0,1]\times[0,2\pi].$$

Applying Stokes's theorem: $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}(\vec{F})\cdot d\vec{S}$

$$\text{Curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \left\langle \frac{\partial z^2}{\partial y} - \frac{\partial x}{\partial z}, -\left(\frac{\partial z^2}{\partial x} - \frac{\partial(-y^2)}{\partial z}\right), \frac{\partial x}{\partial x} - \frac{\partial(-y^2)}{\partial y} \right\rangle$$

$$= \langle 0, 0, 1+2y \rangle = \langle 0, 0, 1+2r\sin(\theta) \rangle$$

$$\therefore \text{Curl}(\vec{F})\cdot(\vec{S}_r \times \vec{S}_\theta) = \begin{vmatrix} 0 & 0 & 1+2r\sin(\theta) \\ \cos(\theta) & \sin(\theta) & -\sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) & -r\cos(\theta) \end{vmatrix}$$

$$= 0 - 0 + (1+2r\sin(\theta)) \begin{vmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{vmatrix} = r + 2r^2\sin(\theta)$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \iint_D \text{Curl}(\vec{F})\cdot(\vec{S}_r \times \vec{S}_\theta)\, dA$$

:) by Stokes's Theorem

$$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} \left(r + 2r^2\sin(\theta)\right) d\theta\, dr$$

$$= \int_{r=0}^{1} \left[r\theta - 2r^2\cos(\theta) \right]_{\theta=0}^{2\pi} dr$$

$$= \int_{r=0}^{1} \left(r(2\pi - 0) - 2r^2(1-1) \right) dr$$

$$= \int_{r=0}^{1} 2\pi r\, dr = \pi r^2 \Big|_{r=0}^{1} = \pi(1-0) = \pi \quad //

Ex: Compute $\iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$ for $\vec{F} = \langle xz, yz, xy \rangle$ and S the part of the sphere $x^2+y^2+z^2=4$ with $z \geq 0$ and $x^2+y^2 \leq 1$.

Sol: Using Stokes's Theorem



$$\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\partial S = \{(x,y,z) : x^2+y^2=1,\ x^2+y^2+z^2=4\}$$
$$= \{(x,y,z) : x^2+y^2=1,\ z=\sqrt{3}\}$$

$$\therefore \vec{r}(\theta) = \langle \cos(\theta), \sin(\theta), \sqrt{3} \rangle \quad \text{on} \quad 0 \leq \theta \leq 2\pi$$

$$\therefore \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

$$= \int_{\theta=0}^{2\pi} \langle \sqrt{3}\cos(\theta), \sqrt{3}\sin(\theta), \cos(\theta)\sin(\theta) \rangle \cdot \langle -\sin(\theta), \cos(\theta), 0 \rangle \, dt$$

$$= \int_{\theta=0}^{2\pi} \left(-\sqrt{3}\cos(\theta)\sin(\theta) + \sqrt{3}\cos(\theta)\sin(\theta) + 0\right) dt$$

$$= \int_{\theta=0}^{2\pi} 0 \, dt$$

$$= 0$$

Useful Observations

Throughout, we assume S is an orientable, piecewise-smooth surface with ∂S a piecewise-smooth, simple, closed curve.

Observe: If S_1 and S_2 have the same boundary curve C, then

$$\iint_{S_1} \text{Curl}(\vec{F}) \cdot d\vec{S} = \int_{\partial S_1 = C = \partial S_2} \vec{F} \cdot dr = \iint_{S_2} \text{Curl}(\vec{F}) \cdot d\vec{S}.$$

Thus we may choose any surface w/ boundary C to apply Stokes...
This is the surfaces version of "Independence of Paths".

Prop: If $f : \mathbb{R}^3 \to \mathbb{R}$, then $\int_{\partial S} \vec{\nabla} f \cdot d\vec{r} = 0$.

pf:
$$\int_{\partial S} \vec{\nabla} f \cdot d\vec{r} = \iint_S \text{Curl}(\vec{\nabla} f) \cdot d\vec{S} = \iint_S \vec{0} \cdot d\vec{S} = 0$$
$\boxtimes$

Prop: If $f, g : \mathbb{R}^3 \to \mathbb{R}$, then $\int_{\partial S} f \nabla g \cdot d\vec{r} = \iint_S (\nabla f \times \nabla g)\, d\vec{S}$.

pf:
$$\int_{\partial S} f \nabla g \cdot d\vec{r} \overset{\text{Stokes's theorem}}{=} \iint_S \text{Curl}(f \nabla g) \cdot d\vec{S}$$

WTS: $\text{Curl}(f \nabla g) = \nabla f \times \nabla g$

$$\text{Curl}(f \nabla g) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f \cdot g_x & f \cdot g_y & f \cdot g_z \end{vmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}[f g_z] - \frac{\partial}{\partial z}[f g_y],\ -\left(\frac{\partial}{\partial x}[f g_z] - \frac{\partial}{\partial z}[f g_x]\right),\ \frac{\partial}{\partial x}[f g_y] - \frac{\partial}{\partial y}[f g_x] \right\rangle$$

$$= \Big\langle (f_y g_z + \cancel{f g_{zy}}) - (f_z g_y + \cancel{f g_{yz}}),\ -\big((f_x g_z + \cancel{f g_{zx}}) - (f_z g_x + \cancel{f g_{xz}})\big),$$
$$(f_x g_y + \cancel{f g_{yx}}) - (f_y g_x - \cancel{f g_{xy}}) \Big\rangle$$

$$= \langle f_y g_z - f_z g_y, -(f_x g_z - f_z g_x), f_x g_y - f_y g_x \rangle$$

$$= \begin{vmatrix} f_y & f_z \\ g_y & g_z \end{vmatrix} \vec{i} - \begin{vmatrix} f_x & f_z \\ g_x & g_z \end{vmatrix} \vec{j} + \begin{vmatrix} f_x & f_y \\ g_x & g_z \end{vmatrix} \vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = \vec{\nabla} f \times \vec{\nabla} g$$

Hence $\displaystyle\int_{\partial S} f \vec{\nabla} g \cdot d\vec{r} = \iint_S \operatorname{curl}((f \vec{\nabla} g)) \cdot ds = \iint_S (\vec{\nabla} f \times \vec{\nabla} g) \cdot d\vec{S}$ $\boxed{//}$

Cor: $\displaystyle\int_{\partial S} f \nabla f \cdot d\vec{r} = \iint_S (\vec{\nabla} f \times \vec{\nabla} f) \cdot d\vec{S} = \iint_S 0 \cdot d\vec{S} = 0$ $\boxed{//}$

Cor: $\displaystyle\int_{\partial S} (f \nabla g + g \nabla f) \cdot d\vec{r} = \iint_S \operatorname{curl}((f \nabla g + g \nabla f)) \cdot d\vec{S}$

$$= \iint_S \left(\operatorname{curl}(f \nabla g) + \operatorname{curl}(g \nabla f)\right) \cdot d\vec{S}$$

$$= \iint_S (\nabla f \times \nabla g + \nabla g \times \nabla f) \cdot d\vec{S}$$

$$= \iint_S (\nabla f \times \nabla g - \nabla f \times \nabla g) \cdot d\vec{S}$$

$$= \iint_S \vec{0} \cdot d\vec{S} = 0$$ $\boxed{//}$